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Analysis of a quasistatic contact problem for piezoelectric materials ☆

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ABSTRACT

We consider a mathematical model which describes the frictional contact between a piezoelectric body and an electrically conductive foundation. The process is quasistatic, the material behavior is modeled with an electro-viscoelastic constitutive law and the contact is described with subdifferential boundary conditions. We derive the variational formulation of the problem which is in the form of a system involving two history-dependent hemivariational inequalities in which the unknowns are the velocity and electric potential field. Then we prove the existence of a unique weak solution to the model. The proof is based on a recent result on history-dependent hemivariational inequalities obtained in Migórski et al. (submitted for publication) [16].

1. Introduction

Piezoelectric materials are real materials characterized by the coupling between mechanical and electrical properties. They are used as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials and piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials.

General models for electro-elastic materials can be found in [2,8,19]. Static frictional contact problems for electro-elastic materials were studied in [3,10,12], under the assumption that the foundation is insulated. Part of these results were extended recently in [14,17] in the case of an electrically conductive foundation. There, the material behavior was described with an electro-elastic constitutive law and the process was assumed to be static. The unique solvability of the corresponding problems was obtained by using arguments of hemivariational inequalities. A quasistatic problem with normal compliance for electro-viscoelastic materials in frictional contact with a conductive foundation was investigated in [9]. There, the variational formulation of the corresponding problem was derived and the existence of a unique weak solution was obtained, under a smallness assumption on the data. The proof was based on arguments of evolutionary variational inequalities with monotone operators and a fixed point theorem. A dynamic contact problem for electro-viscoelastic materials in frictional contact with a conductive foundation was investigated in [15]. There, the weak solvability of the problem was based on arguments on pseudomonotone operators and hemivariational inequalities.

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The present paper represents a continuation of [9,15] and is devoted to the study of a new mathematical model which describes the frictional contact between an electro-viscoelastic body and a conductive foundation. With respect to [9] the novelty of the model consists in the fact that both the contact, the frictional and the conductivity condition are modeled with subdifferential boundary conditions involving nonconvex functionals. With respect to [15] the novelty of the model consists in the fact that here the process is quasistatic and, in addition, the elasticity operator involved in the constitutive law is nonlinear and could be time-dependent. Therefore, the arguments on evolutionary hemivariational inequalities used in [15] do not work in this case. Our interest in this paper is twofold. First, to describe a physical process in which contact, friction and piezoelectric effects are involved and to derive a consistent mathematical model to this process. Second, to illustrate how our recent existence and uniqueness result on history-dependent hemivariational inequalities, obtained in [16], can be used to provide the unique solvability of this mathematical model.

Although in this paper we do not deal with real world applications of our model, we mention that problems involving piezoelectric contact arise in smart structures and various device applications, see for instance [22] and the references therein. For instance, the relative motion of two bodies may be detected by a piezoelectric sensor in frictional contact with them, as stated in [3], and vibration of elastic plates may be obtained by the contact with a piezoelectric actuator under electric voltage, see [21] for details. Also, the contact of a read/write piezoelectric head on a hard disk is based on the mechanical deformation generated by the inverse piezoelectric effect, see for instance [1]; there, error estimates and numerical simulations in the study of the corresponding piezoelectric contact problem are provided.

The rest of paper is structured as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we describe the model of frictional contact between an electro-viscoelastic body and a conductive foundation. In Section 4 we list the assumption on the data and derive a variational formulation of the problem, which is in the form of a system coupling two history-dependent hemivariational inequalities for the displacement and the electric potential, respectively. Then we state our main existence and uniqueness result, Theorem 2. The proof of the theorem is presented in Section 5.

2. Notation and preliminaries

In this section we introduce the notation and some preliminary material which will be used in the next sections. For further details, we refer to [4–6,18].

Given a normed space $(E, \|\cdot\|_E)$ we denote by E^* its dual space and $\langle\cdot,\cdot\rangle_{E^*\times E}$ will represent the duality pairing of E and E^*. The symbol w-E is used for the space E endowed with the weak topology. The space of all linear and continuous operators from a normed space E to a normed space F is denoted by $\mathcal{L}(E,F)$. Let $h : E \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x;v)$, is defined by

$$h^0(x;v) = \limsup_{y\to x,\,\lambda\downarrow 0} \frac{h(y+\lambda v) - h(y)}{\lambda}$$

and the generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \left\{\zeta \in E^* \;\middle|\; h^0(x;v) \geqslant \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\right\}.$$

A locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in E$.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$ and $\Gamma \subseteq \partial\Omega$. Let Y be a closed subspace of $H^1(\Omega;\mathbb{R}^s)$, $s \geqslant 1$, $H = L^2(\Omega;\mathbb{R}^s)$ and $Z = H^\rho(\Omega;\mathbb{R}^s)$ with $\rho \in (1/2, 1)$. Denoting by $i : Y \to Z$ the embedding, by $\gamma : Z \to L^2(\Gamma;\mathbb{R}^s)$ and $\gamma_0 : H^1(\Omega;\mathbb{R}^s) \to H^{1/2}(\Gamma;\mathbb{R}^s) \subset L^2(\Gamma;\mathbb{R}^s)$ the trace operators, we get $\gamma_0 v = \gamma(iv)$ for all $v \in Y$. For simplicity, in what follows we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for all $v \in Y$.

From the theory of Sobolev spaces, we know that (Y, H, Y^*) and (Z, H, Z^*) form evolution triples of spaces and the embedding $Y \subset Z$ is compact. We denote by c_0 the embedding constant of Y into Z, by $\|\gamma\|$ the norm of the trace in $\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^s))$ and by $\gamma^* : L^2(\Gamma;\mathbb{R}^s) \to Z^*$ the adjoint operator to γ. We also introduce the spaces

$$\mathcal{Y} = L^2(0,T;Y), \qquad \mathcal{Z} = L^2(0,T;Z) \quad \text{and} \quad \widehat{\mathcal{H}} = L^2(0,T;H),$$

where $0 < T < +\infty$. Since the embeddings $Y \subseteq Z \subseteq H \subseteq Z^* \subseteq Y^*$ are continuous, it is known that the embeddings $\mathcal{Y} \subseteq \mathcal{Z} \subseteq \widehat{\mathcal{H}} \subseteq \mathcal{Z}^* \subseteq \mathcal{Y}^*$ are also continuous, where $\mathcal{Z}^* = L^2(0,T;Z^*)$ and $\mathcal{Y}^* = L^2(0,T;Y^*)$.

Consider now the operators A and $\mathcal{S}$, and the functions j and f, which satisfy the following conditions.

$$\left.\begin{array}{l}
A : (0,T)\times Y \to Y^* \text{ is such that}\\
\quad \text{(a) } A(\cdot,v) \text{ is measurable on } (0,T) \text{ for all } v \in Y;\\
\quad \text{(b) } A(t,\cdot) \text{ is hemicontinuous and strongly monotone for a.e. } t \in (0,T), \text{ i.e.}\\
\qquad \langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{Y^*\times Y} \geqslant m_1\|v_1 - v_2\|_Y^2 \text{ for all } v_1, v_2 \in Y \text{ with } m_1 > 0;\\
\quad \text{(c) } \|A(t,v)\|_{Y^*} \leqslant a_0(t) + a_1\|v\|_Y \text{ for all } v \in Y, \text{ a.e. } t \in (0,T) \text{ with } a_0 \in L^2(0,T),\\
\qquad a_0 \geqslant 0 \text{ and } a_1 > 0;\\
\quad \text{(d) } A(t,0) = 0 \text{ for a.e. } t \in (0,T).
\end{array}\right\} \tag{1}$$

$$\left.\begin{array}{l} \mathcal{S}:\mathcal{Y}\to\mathcal{Y}^* \text{ is such that} \\ \quad \left\|\mathcal{S}u_1(t)-\mathcal{S}u_2(t)\right\|_{Y^*}\leqslant L_{\mathcal{S}}\displaystyle\int_0^t \left\|u_1(s)-u_2(s)\right\|_Y\,ds \\ \quad \text{for all } u_1,u_2\in\mathcal{Y},\ \text{a.e. } t\in(0,T) \text{ with } L_{\mathcal{S}}>0. \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l} j:\Gamma\times(0,T)\times\mathbb{R}^s\to\mathbb{R} \text{ is such that} \\ \quad \text{(a) } j(\cdot,\cdot,\xi) \text{ is measurable on } \Gamma\times(0,T) \text{ for all } \xi\in\mathbb{R}^s \text{ and there exists } e\in L^2(\Gamma;\mathbb{R}^s) \\ \quad\quad \text{such that } j(\cdot,\cdot,e(\cdot))\in L^1(\Gamma\times(0,T)); \\ \quad \text{(b) } j(x,t,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^s \text{ for a.e. } (x,t)\in\Gamma\times(0,T); \\ \quad \text{(c) } \|\partial j(x,t,\xi)\|_{\mathbb{R}^s}\leqslant\bar{c}_0+\bar{c}_1\|\xi\|_{\mathbb{R}^s} \text{ for a.e. } (x,t)\in\Gamma\times(0,T),\ \text{all } \xi\in\mathbb{R}^s \text{ with } \bar{c}_0,\bar{c}_1\geqslant 0; \\ \quad \text{(d) } (\zeta_1-\zeta_2)\cdot(\xi_1-\xi_2)\geqslant -m_2\|\xi_1-\xi_2\|^2_{\mathbb{R}^s} \text{ for all } \zeta_i,\ \xi_i\in\mathbb{R}^s,\ \zeta_i\in\partial j(x,t,\xi_i), \\ \quad\quad i=1,2,\ \text{a.e. } (x,t)\in\Gamma\times(0,T) \text{ with } m_2\geqslant 0; \\ \quad \text{(e) } j^0(x,t,\xi;-\xi)\leqslant\bar{d}_0\,(1+\|\xi\|_{\mathbb{R}^s}) \text{ for a.e. } (x,t)\in\Gamma\times(0,T),\ \text{all } \xi\in\mathbb{R}^s \text{ with } \bar{d}_0\geqslant 0. \end{array}\right\} \tag{3}$$

$$f\in\mathcal{Y}^*. \tag{4}$$

With these data we consider the problem of finding an element $y\in\mathcal{Y}$ such that

$$\langle A(t,y(t)),z\rangle_{Y^*\times Y}+\langle\mathcal{S}y(t),z\rangle_{Y^*\times Y}+\int_\Gamma j^0(t,\gamma y(t);\gamma z)\,d\Gamma\geqslant\langle f(t),z\rangle_{Y^*\times Y} \tag{5}$$

for all $z\in Y$, a.e. $t\in(0,T)$.

To avoid any confusion, we note that in (5) and below the notation $\mathcal{S}y(t)$ stands for $(\mathcal{S}y)(t)$, i.e. $\mathcal{S}y(t)=(\mathcal{S}y)(t)$ for all $y\in\mathcal{Y}$ and a.e. $t\in(0,T)$. The symbols ∂j and j^0 denote the Clarke subdifferential of a locally Lipschitz function $j(x,t,\cdot)$ and its generalized directional derivative, respectively, cf. Chapter 2.1 of [4].

Note that condition (2) is satisfied for the operator $\mathcal{S}:\mathcal{Y}\to\mathcal{Y}^*$ given by

$$\mathcal{S}y(t)=R\left(t,\int_0^t y(s)\,ds+v_0\right)\quad\text{for all } y\in\mathcal{Y},\ \text{a.e. } t\in(0,T), \tag{6}$$

where $R:(0,T)\times Y\to Y^*$ is such that $R(\cdot,z)$ is measurable on $(0,T)$ for all $z\in Y$, $R(t,\cdot)$ is a Lipschitz continuous operator for a.e. $t\in(0,T)$ and $v_0\in Y$. It is also satisfied for the *Volterra operator* $\mathcal{S}:\mathcal{Y}\to\mathcal{Y}^*$ given by

$$\mathcal{S}y(t)=\int_0^t C(t-s)y(s)\,ds\quad\text{for all } y\in\mathcal{Y},\ \text{a.e. } t\in(0,T), \tag{7}$$

where $C\in L^\infty(0,T;\mathcal{L}(Y,Y^*))$. Clearly, in the case of the operators (6) and (7) the current value $\mathcal{S}y(t)$ at the moment t depends on the history of the values of y at the moments $0\leqslant s\leqslant t$ and, therefore, we refer the operators of the form (6) or (7) as *history-dependent operators*. We extend this definition to all the operators $\mathcal{S}:\mathcal{Y}\to\mathcal{Y}^*$ which satisfy condition (2) and, for this reason, we say that the hemivariational inequalities of the form (5) are *history-dependent hemivariational inequalities*. The main feature of such inequalities consists in the fact that they contain terms which at any moment $t\in(0,T)$ depend on the *history* of the solution up to the moment t. This feature makes the difference with respect to the time-dependent hemivariational inequalities studied in literature in which, usually, the terms involved in are assumed to depend on the *current* value of the solution $y(t)$.

The following existence and uniqueness result for the hemivariational inequality (5) was recently proved in [16].

Theorem 1. *Assume that* (1), (2) *and* (4) *hold. If one of the following hypotheses*

(i) (3)(a)–(d) *and* $m_1>\max\{\sqrt{3}\bar{c}_1,m_2\}c_0^2\|\gamma\|^2$,
(ii) (3) *and* $m_1>m_2c_0^2\|\gamma\|^2$

is satisfied, then inequality (5) *has a solution* $y\in\mathcal{Y}$. *If, in addition, the regularity condition*

either $j(x,t,\cdot)$ *or* $-j(x,t,\cdot)$ *is regular on* $\mathbb{R}^s$ *for a.e.* $(x,t)\in\Gamma\times(0,T)$

holds, then the solution of (5) *is unique.*

3. The model

In this section we describe the problem of frictional contact between a piezoelectric body and a conductive foundation.

The physical setting is as follows. A body made of a piezoelectric material occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we consider a partition of $\partial\Omega$ into three open disjoint parts Γ_D, Γ_N and Γ_C, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$. The body is clamped on Γ_D and therefore the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_b is prescribed on Γ_b. In the reference configuration, the body is in contact over Γ_C with a conductive obstacle, the so called foundation. We assume that the foundation is electrically conductive and its potential is maintained at φ_F. The contact is frictional and there may be electrical charges on the contact surface.

Let $T > 0$, denote by $[0, T]$ the time interval of interest and by $\mathbb{S}^d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$. Then, assuming that the process is quasistatic, the classical model for the contact problem is the following.

Problem $\boldsymbol{P}$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$ and an electric displacement field $\boldsymbol{D} : \Omega \times [0, T] \to \mathbb{R}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}'\big) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{P}^\top \boldsymbol{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{8}$$

$$\boldsymbol{D} = \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{9}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0, T), \tag{10}$$

$$\operatorname{div}\boldsymbol{D} = q_0 \quad \text{in } \Omega \times (0, T), \tag{11}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0, T), \tag{12}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{on } \Gamma_N \times (0, T), \tag{13}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0, T), \tag{14}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b \times (0, T), \tag{15}$$

$$-\sigma_\nu \in \partial j_\nu\big(u'_\nu\big) \quad \text{on } \Gamma_C \times (0, T), \tag{16}$$

$$-\boldsymbol{\sigma}_\tau \in \partial j_\tau\big(\boldsymbol{u}'_\tau\big) \quad \text{on } \Gamma_C \times (0, T), \tag{17}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} \in \partial j_e(\varphi - \varphi_F) \quad \text{on } \Gamma_C \times (0, T), \tag{18}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{19}$$

Note that Problem P represents the quasistatic version of the frictional contact problem considered in [15]. For this reason, we restrict ourselves to provide only a brief explanation of the equations and the conditions (8)–(19) and, for more details, we send the reader to [15].

First, Eqs. (8) and (9) represent the electro-viscoelastic constitutive law, in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, while $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and the elasticity operators, respectively, assumed to be nonlinear and time dependent. Also, $\mathcal{P} = (p_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{P}^\top$ is its transpose, $\boldsymbol{\beta} = (\beta_{ij})$ denotes the electric permittivity tensor and $\boldsymbol{E}(\varphi)$ is the electric field, i.e. $\boldsymbol{E}(\varphi) = -\nabla\varphi = -(\varphi_{,i})$. The tensors $\mathcal{P}$ and $\mathcal{P}^\top$ satisfy the equality $\mathcal{P}\boldsymbol{\varepsilon} \cdot \boldsymbol{\xi} = \boldsymbol{\varepsilon} \cdot \mathcal{P}^\top\boldsymbol{\xi}$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\boldsymbol{\xi} \in \mathbb{R}^d$ and the components of $\mathcal{P}^\top$ are given by $p^\top_{ijk} = p_{kij}$. Note also that, here and below, we assume that the tensors $\mathcal{P}$ and $\boldsymbol{\beta}$ are time-dependent.

Eqs. (10) and (11) represent the balance equation for the stress and the electric-displacement field, respectively. We use them since the process is assumed to be quasistatic. Conditions (12) and (13) are the displacement and traction boundary conditions, whereas (14) and (15) represent the electric boundary conditions. These conditions model the fact that the displacement field and the electrical potential vanish on Γ_D and Γ_a, respectively, while the forces and the electric charges are prescribed on Γ_N and Γ_b, respectively.

The boundary conditions (16), (17), (18) describe the contact, the frictional and the electrical conductivity conditions on the contact surface Γ_C. Here and below the subscripts ν and τ for $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ indicate normal and tangential components of tensors and vectors. The functions j_ν, j_τ and j_e are prescribed, the symbol ∂j_α, $\alpha \in \{\nu, \tau, e\}$, denotes the Clarke subdifferential or the generalized gradient of a locally Lipschitz function j_α and, recall, φ_F represents the electric potential of the foundation, assumed to be given. Concrete examples of frictional models which lead to subdifferential boundary conditions of the form (16), (17) with the functions j_ν and j_τ satisfying assumptions (26) and (27) below can be found in [13]. Here, we only remark that these examples include the viscous contact and the contact with nonmonotone normal damped response, associated to a nonmonotone friction law, to Tresca's friction law or to a power-law friction, see [7,20] for details. Also, examples of frictional models which lead to subdifferential boundary conditions of the form (18) with a function j_e which satisfies assumption (28) below can be found in [15]. When $j_e \equiv 0$, then (18) leads to

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_C \times (0, T).$$

This electrical boundary condition, used in [3,10,12], models the case when the obstacle is a perfect insulator. Note also that the functions j_ν, j_τ and j_e may depend explicitly on the time variable, which allows to model situations when the contact conditions depend on the temperature, which plays the role of a parameter, and which evolution in time is prescribed. Finally, condition (19) represents the initial condition in which $\boldsymbol{u}_0$ denotes the initial displacement.

The system (8)–(19) represents the classical formulation of the electro-viscoelastic frictional contact problem, and by this we mean that the unknowns and the data are smooth functions such that all the derivatives and all the conditions are satisfied in the usual sense, i.e. at each point and at each time instance. However, it is well known that, in general, the classical formulations of contact problems do not have any solution; therefore, in order to provide a result concerning the well posedness of the model, there is a need to reformulate Problem P in a weaker sense, i.e. to derive its variational formulation.

4. Variational formulation and main result

In this section we list the assumptions on the data, we derive a variational formulation of Problem P and state our main existence and uniqueness result, Theorem 2.

Everywhere in this section we use the notation $\boldsymbol{x} = (x_i)$ for a generic point in $\Omega \cup \partial\Omega$. Here and below, unless specified otherwise, the indices i, j, k, l run between 1 and d and the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i/\partial x_j$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d. \end{aligned}$$

In the study of the contact problem (8)–(19) we use standard notation for Lebesgue spaces and Sobolev spaces. We also use the notation $Q = \Omega \times (0,T)$ and $\Sigma_C = \Gamma_C \times (0,T)$. For $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ we still denote by $\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on Γ and we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on $\partial\Omega$ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}.$$

We recall that the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$$

and, if $\boldsymbol{\sigma}$ is sufficiently smooth, then the following Green formula holds

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma \quad \text{for all } \boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d). \tag{20}$$

For the mechanical unknowns of Problem P, we use the spaces

$$\begin{aligned} &V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}, \qquad \mathcal{H} = L^2(\Omega;\mathbb{S}^d),\\ &\mathcal{H}_1 = \{\boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\}. \end{aligned}$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\operatorname{Div}\boldsymbol{\tau}\,dx,$$

and the associated norms $\|\cdot\|_V$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators given by

$$\begin{aligned} &\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \qquad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \text{for all } \boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d),\\ &\operatorname{Div}\boldsymbol{\tau} = (\tau_{ij,j}) \quad \text{for all } \boldsymbol{\tau}\in\mathcal{H}_1. \end{aligned}$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas(\Gamma_D) > 0$, which allows to use Korn's inequality.

For the electric unknowns of Problem P, besides the space $L^2(\Omega;\mathbb{R}^d)$, we use the spaces

$$\begin{aligned} &\Phi = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_a\},\\ &\Phi_1 = \{\boldsymbol{D} = (D_i) \in L^2(\Omega;\mathbb{R}^d) \mid D_{i,i} \in L^2(\Omega)\}. \end{aligned}$$

These are real Hilbert spaces with inner products

$$(\varphi,\psi)_\Phi = \int_\Omega \nabla\varphi\cdot\nabla\psi\,dx, \qquad (\boldsymbol{D},\boldsymbol{F})_{\Phi_1} = \int_\Omega \boldsymbol{D}\cdot\boldsymbol{F}\,dx + \int_\Omega \operatorname{div}\boldsymbol{D}\cdot\operatorname{div}\boldsymbol{F}\,dx.$$

The associated norms are denoted by $\|\cdot\|_\Phi$ and $\|\cdot\|_{\Phi_1}$, respectively. Here ∇ and div are the gradient and divergence operators given by

$$\nabla\psi = (\psi_{,i}) \quad \text{for all } \psi\in\Phi, \qquad \operatorname{div}\boldsymbol{D} = D_{i,i} \quad \text{for all } \boldsymbol{D}\in\Phi_1.$$

Completeness of the space $(\Phi,\|\cdot\|_\Phi)$ is a consequence of the assumption $meas(\Gamma_a)>0$ which allows to use Friedrichs–Poincaré's inequality. Recall also that for $\boldsymbol{D}\in\Phi_1$, $\boldsymbol{D}\cdot\boldsymbol{\nu}\in H^{-1/2}(\Gamma)$ is well defined. Moreover, if $\boldsymbol{D}$ is sufficiently regular, then

$$\int_\Omega \boldsymbol{D}\cdot\nabla\psi\,dx + \int_\Omega \operatorname{div}\boldsymbol{D}\,\psi\,dx = \int_{\partial\Omega} \boldsymbol{D}\cdot\boldsymbol{\nu}\,\psi\,d\Gamma \quad \text{for all } \psi\in H^1(\Omega). \tag{21}$$

We now list the assumptions on the data. We assume that the viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{B}$ satisfy

$$\left.\begin{array}{l}
\mathcal{A}: Q\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\
\quad \text{(a) } \mathcal{A}(\cdot,\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } Q \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d;\\
\quad \text{(b) } \mathcal{A}(\boldsymbol{x},t,\cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } (\boldsymbol{x},t)\in Q;\\
\quad \text{(c) } (\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d}\\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t)\in Q \text{ with } m_{\mathcal{A}}>0;\\
\quad \text{(d) } \|\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon})\|_{\mathbb{S}^d}\leqslant \bar{a}_0(\boldsymbol{x},t)+\bar{a}_1\|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d,\\
\qquad\quad \text{a.e. } (\boldsymbol{x},t)\in Q \text{ with } \bar{a}_0\in L^2(Q), \bar{a}_0\geqslant 0 \text{ and } \bar{a}_1>0;\\
\quad \text{(e) } \mathcal{A}(\boldsymbol{x},t,\boldsymbol{0})=\boldsymbol{0} \text{ for a.e. } (\boldsymbol{x},t)\in Q.
\end{array}\right\} \tag{22}$$

$$\left.\begin{array}{l}
\mathcal{B}: Q\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\
\quad \text{(a) } \mathcal{B}(\cdot,\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } Q \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d;\\
\quad \text{(b) } \|\mathcal{B}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1)-\mathcal{B}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d}\leqslant L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}\\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t)\in Q \text{ with } L_{\mathcal{B}}>0;\\
\quad \text{(c) } \mathcal{B}(\cdot,\cdot,\boldsymbol{0})\in L^2(Q;\mathbb{S}^d).
\end{array}\right\} \tag{23}$$

The piezoelectric tensor $\mathcal{P}$ and the electric permittivity tensor $\boldsymbol{\beta}$ satisfy

$$\left.\begin{array}{l}
\mathcal{P}: Q\times\mathbb{S}^d\to\mathbb{R}^d \text{ is such that}\\
\quad \text{(a) } \mathcal{P}(\boldsymbol{x},t,\boldsymbol{\varepsilon})=p(\boldsymbol{x},t)\boldsymbol{\varepsilon} \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d, \text{ a.e. } (\boldsymbol{x},t)\in Q;\\
\quad \text{(b) } p(\boldsymbol{x},t)=(p_{ijk}(\boldsymbol{x},t)) \text{ with } p_{ijk}\in L^\infty(Q).
\end{array}\right\} \tag{24}$$

$$\left.\begin{array}{l}
\boldsymbol{\beta}: Q\times\mathbb{R}^d\to\mathbb{R}^d \text{ is such that}\\
\quad \text{(a) } \boldsymbol{\beta}(\boldsymbol{x},t,\boldsymbol{\xi})=\boldsymbol{\beta}(\boldsymbol{x},t)\boldsymbol{\xi} \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^d, \text{ a.e. } (\boldsymbol{x},t)\in Q;\\
\quad \text{(b) } \boldsymbol{\beta}(\boldsymbol{x},t)=(\beta_{ij}(\boldsymbol{x},t)) \text{ with } \beta_{ij}\in L^\infty(Q);\\
\quad \text{(c) } \beta_{ij}(\boldsymbol{x},t)\xi_i\xi_j\geqslant m_\beta\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d} \text{ for all } \boldsymbol{\xi}=(\xi_i)\in\mathbb{R}^d,\\
\qquad\quad \text{a.e. } (\boldsymbol{x},t)\in Q \text{ with } m_\beta>0.
\end{array}\right\} \tag{25}$$

The contact potentials j_ν, j_τ and j_e satisfy the following hypotheses.

$$\left.\begin{array}{l}
j_\nu: \Sigma_C\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\
\quad \text{(a) } j_\nu(\cdot,\cdot,r) \text{ is measurable on } \Sigma_C \text{ for all } r\in\mathbb{R} \text{ and there}\\
\qquad\quad \text{exists } e_1\in L^2(\Gamma_C) \text{ such that } j_\nu(\cdot,\cdot,e_1(\cdot))\in L^1(\Sigma_C);\\
\quad \text{(b) } j_\nu(\boldsymbol{x},t,\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } (\boldsymbol{x},t)\in\Sigma_C;\\
\quad \text{(c) } |\partial j_\nu(\boldsymbol{x},t,r)|\leqslant c_{0\nu}+c_{1\nu}|r| \text{ for all } r\in\mathbb{R}, \text{ a.e. } (\boldsymbol{x},t)\in\Sigma_C\\
\qquad\quad \text{with } c_{0\nu},c_{1\nu}\geqslant 0;\\
\quad \text{(d) } (\zeta_1-\zeta_2)(r_1-r_2)\geqslant -m_\nu|r_1-r_2|^2 \text{ for all } \zeta_i\in\partial j_\nu(\boldsymbol{x},t,r_i),\\
\qquad\quad r_i\in\mathbb{R},\ i=1,2, \text{ a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ with } m_\nu\geqslant 0;\\
\quad \text{(e) } j^0_\nu(\boldsymbol{x},t,r;-r)\leqslant d_\nu(1+|r|) \text{ for all } r\in\mathbb{R}, \text{ a.e. } (\boldsymbol{x},t)\in\Sigma_C\\
\qquad\quad \text{with } d_\nu\geqslant 0.
\end{array}\right\} \tag{26}$$

$$\left.\begin{array}{l} j_\tau : \Sigma_C \times \mathbb{R}^d \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\tau(\cdot,\cdot,\boldsymbol{\xi}) \text{ is measurable on } \Sigma_C \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and there} \\ \qquad \text{exists } \mathbf{e}_2 \in L^2(\Gamma_C;\mathbb{R}^d) \text{ such that } j_\tau(\cdot,\cdot,(\mathbf{e}_2(\cdot))_\tau) \in L^1(\Sigma_C); \\ \quad \text{(b) } j_\tau(\boldsymbol{x},t,\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } (\boldsymbol{x},t) \in \Sigma_C; \\ \quad \text{(c) } \|\partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi})\|_{\mathbb{R}^d} \leqslant c_{0\tau} + c_{1\tau}\|\boldsymbol{\xi}\|_{\mathbb{R}^d} \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \\ \qquad \text{a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } c_{0\tau}, c_{1\tau} \geqslant 0; \\ \quad \text{(d) } (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2)\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geqslant -m_\tau\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^d} \text{ for all} \\ \qquad \boldsymbol{\zeta}_i \in \partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_i),\ \boldsymbol{\xi}_i \in \mathbb{R}^d,\ i=1,2, \text{ a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } m_\tau \geqslant 0; \\ \quad \text{(e) } j^0_\tau(\boldsymbol{x},t,\boldsymbol{\xi};-\boldsymbol{\xi}) \leqslant d_\tau(1+\|\boldsymbol{\xi}\|_{\mathbb{R}^d}) \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \\ \qquad \text{a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } d_\tau \geqslant 0. \end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l} j_e : \Sigma_C \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_e(\cdot,\cdot,r) \text{ is measurable on } \Sigma_C \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } e_3 \in L^2(\Gamma_C) \text{ such that } j_e(\cdot,\cdot,e_3(\cdot)) \in L^1(\Sigma_C); \\ \quad \text{(b) } j_e(\boldsymbol{x},t,\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } (\boldsymbol{x},t) \in \Sigma_C; \\ \quad \text{(c) } |\partial j_e(\boldsymbol{x},t,r)| \leqslant c_{0e} + c_{1e}|r| \text{ for all } r \in \mathbb{R}, \text{ a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } c_{0e}, c_{1e} \geqslant 0; \\ \quad \text{(d) } (\zeta_1 - \zeta_2)(r_1 - r_2) \geqslant -m_e|r_1 - r_2|^2 \text{ for all } \zeta_i \in \partial j_e(\boldsymbol{x},t,r_i), \\ \qquad r_i \in \mathbb{R},\ i=1,2, \text{ a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } m_e \geqslant 0; \\ \quad \text{(e) } j^0_e(\boldsymbol{x},t,r;-r) \leqslant d_e(1+|r|) \text{ for all } r \in \mathbb{R}, \text{ a.e. } (\boldsymbol{x},t) \in \Sigma_C \text{ with } d_e \geqslant 0. \end{array}\right\} \tag{28}$$

We also assume that the body forces, tractions, volume and surface electric charge densities have the regularity

$$\boldsymbol{f}_0 \in L^2\big(0,T;L^2\big(\Omega;\mathbb{R}^d\big)\big), \qquad \boldsymbol{f}_N \in L^2\big(0,T;L^2\big(\Gamma_N;\mathbb{R}^d\big)\big), \tag{29}$$

$$q_0 \in L^2\big(0,T;L^2(\Omega)\big), \qquad q_b \in L^2\big(0,T;L^2(\Gamma_b)\big). \tag{30}$$

Finally, the potential of the foundation and the initial displacement are such that

$$\varphi_F \in L^\infty(\Gamma_C), \tag{31}$$

$$\boldsymbol{u}_0 \in V. \tag{32}$$

We turn now to the variational formulation of Problem P. To this end we suppose that $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D})$ is a quadruple of sufficiently smooth functions which solve (8)–(19). Let $\boldsymbol{v} \in V$ and $\psi \in \Phi$. Then, using (10) and (20), we have

$$\big(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} = \big(\boldsymbol{f}_0(t),\boldsymbol{v}\big)_{L^2(\Omega;\mathbb{R}^d)} + \int_{\partial\Omega} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma \quad \text{for a.e. } t \in (0,T). \tag{33}$$

We take into account the boundary conditions (12) and (13) to see that

$$\int_{\partial\Omega} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma = \int_{\Gamma_N} \boldsymbol{f}_N(t)\cdot\boldsymbol{v}\,d\Gamma + \int_{\Gamma_C} \big(\sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau\big)\,d\Gamma \tag{34}$$

for a.e. $t \in (0,T)$. On the other hand, from the definition of the Clarke subdifferential, using (16) and (17), we have

$$-\sigma_\nu(t)v_\nu \leqslant j^0_\nu\big(t,u'_\nu(t);v_\nu\big), \qquad -\boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau \leqslant j^0_\tau\big(t,\boldsymbol{u}'_\tau(t);\boldsymbol{v}_\tau\big) \quad \text{on } \Sigma_C,$$

which imply that

$$\int_{\Gamma_C} \big(\sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau\big)\,d\Gamma \geqslant -\int_{\Gamma_C} \big(j^0_\nu\big(t,u'_\nu(t);v_\nu\big) + j^0_\tau\big(t,\boldsymbol{u}'_\tau(t);\boldsymbol{v}_\tau\big)\big)\,d\Gamma \tag{35}$$

for a.e. $t \in (0,T)$. Consider the function $\boldsymbol{f} : (0,T) \to V^*$ given by

$$\big\langle \boldsymbol{f}(t),\boldsymbol{v}\big\rangle_{V^*\times V} = \big(\boldsymbol{f}_0(t),\boldsymbol{v}\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(\boldsymbol{f}_N(t),\boldsymbol{v}\big)_{L^2(\Gamma_N;\mathbb{R}^d)} \tag{36}$$

for all $\boldsymbol{v} \in V$ and a.e. $t \in (0,T)$. We combine (33)–(36) to obtain

$$\big(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + \int_{\Gamma_C} \big(j^0_\nu\big(t,u'_\nu(t);v_\nu\big) + j^0_\tau\big(t,\boldsymbol{u}'_\tau(t);\boldsymbol{v}_\tau\big)\big)\,d\Gamma \geqslant \big\langle \boldsymbol{f}(t),\boldsymbol{v}\big\rangle_{V^*\times V} \tag{37}$$

for a.e. $t \in (0,T)$. Similarly, for every $\psi \in \Phi$, from (11) and (21) we deduce that

$$(\boldsymbol{D},\nabla\psi)_{L^2(\Omega;\mathbb{R}^d)} + \int_\Omega q_0(t)\psi\,dx = \int_{\partial\Omega} \boldsymbol{D}\cdot\boldsymbol{\nu}\psi\,d\Gamma \quad \text{for a.e. } t \in (0,T)$$

and by (14), (15), we get

$$-(\boldsymbol{D}, \nabla\psi)_{L^2(\Omega;\mathbb{R}^d)} + \int_{\Gamma_C} \boldsymbol{D}\cdot\boldsymbol{\nu}\psi\, d\Gamma = \langle q(t), \psi\rangle_{\Phi^*\times\Phi} \quad \text{for a.e. } t \in (0,T), \tag{38}$$

where $q : (0,T) \to \Phi^*$ is the function given by

$$\langle q(t), \psi\rangle_{\Phi^*\times\Phi} = \int_{\Omega} q_0(t)\psi\, dx - \int_{\Gamma_b} q_b(t)\psi\, d\Gamma$$

for all $\psi \in \Phi$ and a.e. $t \in (0,T)$. From the definition of the Clarke subdifferential and (18), we have

$$\boldsymbol{D}\cdot\boldsymbol{\nu}\psi \leqslant j_e^0(\varphi - \varphi_F; \psi) \quad \text{on } \Sigma_C,$$

which implies that

$$\int_{\Gamma_C} \boldsymbol{D}\cdot\boldsymbol{\nu}\psi\, d\Gamma \leqslant \int_{\Gamma_C} j_e^0(\varphi - \varphi_F; \psi)\, d\Gamma \quad \text{for a.e. } t \in (0,T). \tag{39}$$

We combine now (38) and (39) to obtain

$$-(\boldsymbol{D}, \nabla\psi)_{L^2(\Omega;\mathbb{R}^d)} + \int_{\Gamma_C} j_e^0(\varphi - \varphi_F; \psi)\, d\Gamma \geqslant \langle q(t), \psi\rangle_{\Phi^*\times\Phi} \quad \text{for a.e. } t \in (0,T). \tag{40}$$

Let $\boldsymbol{w} = \boldsymbol{u}'$ denote the velocity field. Then, by using the initial condition (19), it follows that

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0 \quad \text{for all } t \in [0,T]. \tag{41}$$

We substitute now (8) in (37), (9) in (40), use the equality $\boldsymbol{E}(\varphi) = -\nabla\varphi$ and equality (41) to derive the following variational formulation of Problem P, in terms of displacement and electric potential fields.

Problem P_V. *Find a velocity field $\boldsymbol{w} : [0,T] \to V$ and an electric potential $\varphi : [0,T] \to \Phi$ such that*

$$\begin{aligned}
&\big(\mathcal{A}\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{w}(t)\big)\big), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + \left(\mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0\right)\right), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} \\
&\quad + \big(\mathcal{P}^\top\big(t, \nabla\varphi(t)\big), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} + \int_{\Gamma_C} \big(j_\nu^0\big(t, w_\nu(t); v_\nu\big) + j_\tau^0\big(t, \boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau\big)\big)\, d\Gamma \\
&\quad \geqslant \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} \quad \textit{for all } \boldsymbol{v} \in V,\ \textit{a.e. } t \in (0,T),
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\big(\boldsymbol{\beta}\big(t, \nabla\varphi(t)\big), \nabla\psi\big)_{L^2(\Omega;\mathbb{R}^d)} - \left(\mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0\right)\right), \nabla\psi\right)_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad + \int_{\Gamma_C} j_e^0\big(\varphi(t) - \varphi_F; \psi\big)\, d\Gamma \geqslant \langle q(t), \psi\rangle_{\Phi^*\times\Phi} \quad \textit{for all } \psi \in \Phi, \textit{a.e. } t \in (0,T).
\end{aligned} \tag{43}$$

We denote by $\|\mathcal{P}\|$ the norm of the operator $\mathcal{P}$ defined by

$$\|\mathcal{P}\| = \max_{i,j,k} \|p_{ijk}\|_{L^\infty(Q)}.$$

Our main result in the study of Problem P_V that we state here and prove in the next section is the following.

Theorem 2. *Assume that* (22)–(25) *and* (29)–(32) *hold, one of the following hypotheses*

(i) (26)(a)–(d), (27)(a)–(d), (28)(a)–(d) *and*

$$\min\{m_{\mathcal{A}}, m_\beta\} - \frac{\|\mathcal{P}\|}{2} > \max\{\sqrt{3}c_{1\nu}, \sqrt{3}c_{1\tau}, \sqrt{3}c_{1e}, m_\nu, m_\tau, m_e\}c_0^2\|\gamma\|^2,$$

(ii) (26), (27), (28) *and* $\min\{m_{\mathcal{A}}, m_\beta\} - \dfrac{\|\mathcal{P}\|}{2} > \max\{m_\nu, m_\tau, m_e\}c_0^2\|\gamma\|^2$

is satisfied and

$$\left.\begin{array}{l}\text{either } j_\nu(\boldsymbol{x},t,\cdot),\ j_\tau(\boldsymbol{x},t,\cdot) \text{ and } j_e(\boldsymbol{x},t,\cdot) \text{ are regular}\\ \text{or } -j_\nu(\boldsymbol{x},t,\cdot),\ -j_\tau(\boldsymbol{x},t,\cdot) \text{ and } -j_e(\boldsymbol{x},t,\cdot) \text{ are regular}\end{array}\right\} \tag{44}$$

for a.e. $(\boldsymbol{x},t)\in\Sigma_C$. *Then Problem* P_V *has a unique solution.*

Let $(\boldsymbol{w},\varphi)$ be a solution of Problem P_V and denote by $\boldsymbol{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ the functions defined by (41), (8) and (9), respectively. Then, the quadruple $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D})$ is called a *weak solution* of the frictional contact problem (8)–(19). We conclude that, under the hypotheses of Theorem 2, the frictional contact problem (8)–(19) has a unique weak solution and it is easy to see that solution satisfies

$$\boldsymbol{u}\in W^{1,2}(0,T;V),\qquad \boldsymbol{\sigma}\in L^2(0,T;\mathcal{H}_1),\qquad \varphi\in L^2(0,T;\Phi),\qquad \boldsymbol{D}\in L^2(0,T;\Phi_1).$$

5. Proof of Theorem 2

The proof is based on the result of Theorem 1 in which $\Gamma=\Gamma_C\subset\partial\Omega$ and $Y=V\times\Phi$. The product Hilbert space Y is endowed with the inner product

$$(y,z)_Y=(\boldsymbol{w},\boldsymbol{v})_V+(\varphi,\psi)_\Phi\quad\text{for } y,z\in Y,\qquad y=(\boldsymbol{w},\varphi),\qquad z=(\boldsymbol{v},\psi)$$

and the associated norm $\|\cdot\|_Y$. Also, let $\mathcal{Y}=L^2(0,T;Y)$. In what follows, we associate with Problem P_V, the operators A and $\mathcal{S}$, and the functions j and f, then we show that they satisfy the hypotheses of Theorem 1.

First, we consider the operator $A:(0,T)\times Y\to Y^*$ defined by

$$\langle A(t,y),z\rangle_{Y^*\times Y}=\big(\mathcal{A}\big(t,\boldsymbol{\varepsilon}(\boldsymbol{w})\big),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}+\big(\boldsymbol{\beta}(t,\nabla\varphi),\nabla\psi\big)_{L^2(\Omega;\mathbb{R}^d)}+\big(\mathcal{P}^\top(t,\nabla\varphi),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}} \tag{45}$$

for all $y=(\boldsymbol{w},\varphi)\in Y$, $z=(\boldsymbol{v},\psi)\in Y$, a.e. $t\in(0,T)$.

We prove that A satisfies the hypothesis (1). The condition (1)(d) is obvious and (1)(a) follows from Fubini's theorem. Using (22), (24), (25) and the Hölder inequality, we have

$$\begin{aligned}\big|\langle A(t,y),z\rangle_{Y^*\times Y}\big|&\leqslant\int_\Omega\big(\big\|\mathcal{A}\big(t,\boldsymbol{\varepsilon}(\boldsymbol{w})\big)\big\|_{\mathbb{S}^d}\big\|\boldsymbol{\varepsilon}(\boldsymbol{v})\big\|_{\mathbb{S}^d}+\big\|\boldsymbol{\beta}(t,\nabla\varphi)\big\|_{\mathbb{R}^d}\|\nabla\psi\|_{\mathbb{R}^d}+\big\|\mathcal{P}^\top(t,\nabla\varphi)\big\|_{\mathbb{S}^d}\big\|\boldsymbol{\varepsilon}(\boldsymbol{v})\big\|_{\mathbb{S}^d}\big)\,dx\\&\leqslant\big(\big\|\bar a_0(t)\big\|_{L^2(\Omega)}+\bar a_1\|\boldsymbol{w}\|_V+\big\|\mathcal{P}^\top\big\|\|\varphi\|_\Phi\big)\|\boldsymbol{v}\|_V+\|\boldsymbol{\beta}\|\|\varphi\|_\Phi\|\psi\|_\Phi\leqslant\big(a_0(t)+a_1\|y\|_Y\big)\|z\|_Y\end{aligned}$$

for all $y=(\boldsymbol{w},\varphi)$, $z=(\boldsymbol{v},\psi)$, y, $z\in Y$, a.e. $t\in(0,T)$ with $\|\boldsymbol{\beta}\|=\max_{i,j}\|\beta_{ij}\|_{L^\infty(Q)}$, $a_0(t)=\|\bar a_0(t)\|_{L^2(\Omega)}$ and $a_1>0$. This gives $\|A(t,y)\|_{Y^*}\leqslant a_0(t)+a_1\|y\|_Y$ for all $y\in Y$, a.e. $t\in(0,T)$ and yields the boundedness condition (1)(c).

We now observe that the inequalities in the hypotheses (i) or (ii) imply

$$\|\mathcal{P}\|<2\min\{m_{\mathcal{A}},m_\beta\}. \tag{46}$$

We claim that under the smallness assumption (46), the operator $A(t,\cdot)$ is strongly monotone for a.e. $t\in(0,T)$. Indeed, let $y_1=(\boldsymbol{w}_1,\varphi_1)$, $y_2=(\boldsymbol{w}_2,\varphi_2)\in Y$ and $t\in(0,T)\setminus\mathcal{N}$ with $meas(\mathcal{N})=0$. It follows from (45), (22) and (25) that

$$\begin{aligned}\langle A(t,y_1)-A(t,y_2),y_1-y_2\rangle_{Y^*\times Y}&\geqslant m_{\mathcal{A}}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V^2+m_\beta\|\varphi_1-\varphi_2\|_\Phi^2\\&\quad-\|\mathcal{P}\|\|\varphi_1-\varphi_2\|_\Phi\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V\quad\text{a.e. } t\in(0,T).\end{aligned}$$

Therefore, by (46) and the inequality $ab\leqslant\frac{1}{2}(a^2+b^2)$ for a, $b\in\mathbb{R}$, we obtain

$$\begin{aligned}\langle A(t,y_1)-A(t,y_2),y_1-y_2\rangle_{Y^*\times Y}&\geqslant\left(m_{\mathcal{A}}-\frac{1}{2}\|\mathcal{P}\|\right)\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V^2+\left(m_\beta-\frac{1}{2}\|\mathcal{P}\|\right)\|\varphi_1-\varphi_2\|_\Phi^2\\&\geqslant m_1\|y_1-y_2\|_Y^2\end{aligned}$$

for a.e. $t\in(0,T)$ with $m_1=\min\{m_{\mathcal{A}},m_\beta\}-\frac{\|\mathcal{P}\|}{2}>0$. Hence we deduce the strong monotonicity of $A(t,\cdot)$ for a.e. $t\in(0,T)$.

In order to show that the operator A satisfies (1)(b), it suffices to establish the continuity of $A(t,\cdot)$ for a.e. $t\in(0,T)$. To this end, let $t\in(0,T)\setminus\mathcal{N}$ with $meas(\mathcal{N})=0$ and $\{y_n\}\subset Y$, $y_n\to y$ in Y, as $n\to\infty$. Denoting $y_n=(\boldsymbol{w}_n,\varphi_n)$ and $y=(\boldsymbol{w},\varphi)$, we have $\boldsymbol{w}_n\to\boldsymbol{w}$ in V and $\varphi_n\to\varphi$ in Φ. This entails that $\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\to\boldsymbol{\varepsilon}(\boldsymbol{w})$ in $L^2(\Omega;\mathbb{S}^d)$ and $\nabla\varphi_n\to\nabla\varphi$ in $L^2(\Omega;\mathbb{R}^d)$. Passing to a subsequence, if necessary, for a subsequence (not relabeled), we have

$$\boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\to\boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\quad\text{in }\mathbb{S}^d,\qquad\nabla\varphi_n(\boldsymbol{x})\to\nabla\varphi(\boldsymbol{x})\quad\text{in }\mathbb{R}^d \tag{47}$$

with

$$\big\|\boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\big\|_{\mathbb{S}^d}\leqslant\eta(\boldsymbol{x}),\qquad\big\|\nabla\varphi_n(\boldsymbol{x})\big\|_{\mathbb{R}^d}\leqslant\eta(\boldsymbol{x})$$

for a.e. $\boldsymbol{x} \in \Omega$, $\eta \in L^2(\Omega)$. Because $\mathcal{A}(\boldsymbol{x}, t, \cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $(\boldsymbol{x}, t) \in Q$, we obtain

$$\mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\big) \to \mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\big) \quad \text{in } \mathbb{S}^d, \text{ a.e. } (\boldsymbol{x}, t) \in Q.$$

Also, from (22)(d), we get

$$\begin{aligned}
&\big\|\mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\big) - \mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\big)\big\|_{\mathbb{S}^d}^2 \\
&\quad \leqslant 2\big(\bar{a}_0(\boldsymbol{x}, t) + \bar{a}_1 \|\boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\|_{\mathbb{S}^d}\big)^2 + 2\big(\bar{a}_0(\boldsymbol{x}, t) + \bar{a}_1 \|\boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\|_{\mathbb{S}^d}\big)^2 \\
&\quad \leqslant 8\bar{a}_0^2(\boldsymbol{x}, t) + 4\bar{a}_1\big(\eta^2(\boldsymbol{x}) + \|\boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\|_{\mathbb{S}^d}^2\big)
\end{aligned}$$

for a.e. $(\boldsymbol{x}, t) \in Q$. By the Lebesgue dominated convergence theorem, we obtain

$$\big\|\mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\big) - \mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w})\big)\big\|_{\mathcal{H}}^2 = \int_\Omega \big\|\mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)(\boldsymbol{x})\big) - \mathcal{A}\big(\boldsymbol{x}, t, \boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x})\big)\big\|_{\mathbb{S}^d}^2 \, dx \to 0.$$

Similarly, from (24), (25) and (47), we deduce the convergences

$$\begin{aligned}
&\mathcal{P}^\top(t, \nabla\varphi_n) \to \mathcal{P}^\top(t, \nabla\varphi) \quad \text{in } \mathcal{H}, \\
&\boldsymbol{\beta}(t, \nabla\varphi_n) \to \boldsymbol{\beta}(t, \nabla\varphi) \quad \text{in } L^2\big(\Omega; \mathbb{R}^d\big).
\end{aligned}$$

From the following inequality

$$\begin{aligned}
\langle A(t, y_n) - A(t, y), z\rangle_{Y^*\times Y} &\leqslant \big|\big(\mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\big) - \mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w})\big), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}\big| \\
&\quad + \big|\big(\mathcal{P}^\top(t, \nabla\varphi_n - \nabla\varphi), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}\big| + \big|\big(\boldsymbol{\beta}(t, \nabla\varphi_n - \nabla\varphi), \nabla\psi\big)_{L^2(\Omega;\mathbb{R}^d)}\big| \\
&\leqslant \big(\big\|\mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\big) - \mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{w})\big)\big\|_{\mathcal{H}} + \big\|\mathcal{P}^\top(t, \nabla\varphi_n - \nabla\varphi)\big\|_{\mathcal{H}} \\
&\quad + \big\|\boldsymbol{\beta}(t, \nabla\varphi_n - \nabla\varphi)\big\|_{L^2(\Omega;\mathbb{R}^d)}\big)\|z\|_Y
\end{aligned}$$

for every $z = (\boldsymbol{v}, \psi) \in Y$ and the convergences obtained above, we conclude that $A(t, y_n) \to A(t, y)$ in Y^* for a.e. $t \in (0, T)$. This implies that $A(t, \cdot)$ is continuous for a.e. $t \in (0, T)$. We conclude that the operator A given by (45) satisfies (1).

Next, we define the operator $\mathcal{S} : \mathcal{Y} \to \mathcal{Y}^*$ by

$$\langle \mathcal{S}y(t), z\rangle_{Y^*\times Y} = \left(\mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0\right)\right), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} - \left(\mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}(s)\, ds + \boldsymbol{u}_0\right)\right), \nabla\psi\right)_{L^2(\Omega;\mathbb{R}^d)} \tag{48}$$

for all $y = (\boldsymbol{w}, \varphi) \in \mathcal{Y}$, $z = (\boldsymbol{v}, \psi) \in Y$, a.e. $t \in (0, T)$. From (23), (24) and (32), we can assert that the operator $\mathcal{S}$ given by (48) satisfies the condition in the hypothesis (2). In fact, for $y_1 = (\boldsymbol{w}_1, \varphi_1)$, $y_2 = (\boldsymbol{w}_2, \varphi_2) \in \mathcal{Y}$, $z = (\boldsymbol{v}, \psi) \in Y$ and a.e. $t \in (0, T)$, we have

$$\begin{aligned}
&\langle \mathcal{S}y_1(t) - \mathcal{S}y_2(t), z\rangle_{Y^*\times Y} \\
&\quad = \left(\mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_1(s)\, ds + \boldsymbol{u}_0\right)\right) - \mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_2(s)\, ds + \boldsymbol{u}_0\right)\right), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} \\
&\qquad - \mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_1(s)\, ds + \boldsymbol{u}_0\right)\right) + \left(\mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_2(s)\, ds + \boldsymbol{u}_0\right)\right), \nabla\psi\right)_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad \leqslant \left\|\mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_1(s)\, ds + \boldsymbol{u}_0\right)\right) - \mathcal{B}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_2(s)\, ds + \boldsymbol{u}_0\right)\right)\right\|_{\mathcal{H}} \|z\|_Y \\
&\qquad + \left\|\mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_1(s)\, ds + \boldsymbol{u}_0\right)\right) - \mathcal{P}\left(t, \boldsymbol{\varepsilon}\left(\int_0^t \boldsymbol{w}_2(s)\, ds + \boldsymbol{u}_0\right)\right)\right\|_{L^2(\Omega;\mathbb{R}^d)} \|z\|_Y.
\end{aligned}$$

Hence, by (23) and (24), we obtain

$$
\begin{aligned}
&\left\|\mathcal{S}y_1(t)-\mathcal{S}y_2(t)\right\|_{Y^*}\\
&\quad\leqslant\left\|\mathcal{B}\left(t,\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_1(s)\,ds+\boldsymbol{u}_0\right)\right)-\mathcal{B}\left(t,\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_2(s)\,ds+\boldsymbol{u}_0\right)\right)\right\|_{\mathcal{H}}\\
&\qquad+\left\|\mathcal{P}\left(t,\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_1(s)\,ds+\boldsymbol{u}_0\right)\right)-\mathcal{P}\left(t,\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_2(s)\,ds+\boldsymbol{u}_0\right)\right)\right\|_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad\leqslant L_{\mathcal{S}}\left\|\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_1(s)\,ds+\boldsymbol{u}_0\right)-\boldsymbol{\varepsilon}\left(\int_0^t\boldsymbol{w}_2(s)\,ds+\boldsymbol{u}_0\right)\right\|_{\mathcal{H}}\\
&\quad\leqslant L_{\mathcal{S}}\int_0^t\left\|y_1(s)-y_2(s)\right\|_Y\,ds,
\end{aligned}
$$

where $L_{\mathcal{S}}=2\max\{L_{\mathcal{B}},\|\mathcal{P}\|\}>0$. Subsequently, from (23), it is easily seen that

$$\left\|\mathcal{B}(\boldsymbol{x},t,\boldsymbol{\varepsilon})\right\|_{\mathbb{S}^d}\leqslant b(\boldsymbol{x},t)+L_{\mathcal{B}}\|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d}\quad\text{for all }\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \text{a.e. }(\boldsymbol{x},t)\in Q,$$

where $b(\boldsymbol{x},t)=\|\mathcal{B}(\boldsymbol{x},t,\mathbf{0})\|_{\mathbb{S}^d}$, $b\in L^2(Q)$, $b\geqslant 0$. Let $y\in\mathcal{Y}$. Then, from the inequality

$$\left\|\mathcal{S}y(t)\right\|_{Y^*}\leqslant L_{\mathcal{S}}\|y\|_{L^1(0,t;Y)}+\left\|\mathcal{S}\mathbf{0}(t)\right\|_{Y^*}\leqslant L_{\mathcal{S}}\|y\|_{L^1(0,t;Y)}+\left\|\mathcal{B}\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\big)\right\|_{\mathcal{H}}+\|\mathcal{P}\|\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\right\|_{L^2(\Omega;\mathbb{R}^d)}$$

it follows that $\|\mathcal{S}y\|_{\mathcal{Y}^*}\leqslant c(\|b\|_{L^2(Q)}+\|\boldsymbol{u}_0\|_V+\|\mathcal{P}\|+\|y\|_{\mathcal{Y}})$ with $c>0$. Therefore $\mathcal{S}$ is well defined and takes values in $\mathcal{Y}^*$. From the above we deduce that the operator $\mathcal{S}$ given by (48) satisfies the hypothesis (2).

We also consider the function $j:\Sigma_C\times\mathbb{R}^{d+1}\to\mathbb{R}$ defined by

$$j(\boldsymbol{x},t,\boldsymbol{\xi},r)=\sum_{i=1}^{3}j_i(\boldsymbol{x},t,\boldsymbol{\xi},r),\tag{49}$$

where

$$
\begin{aligned}
&j_1(\boldsymbol{x},t,\boldsymbol{\xi},r)=j_\nu(\boldsymbol{x},t,\xi_\nu),\\
&j_2(\boldsymbol{x},t,\boldsymbol{\xi},r)=j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_\tau),\\
&j_3(\boldsymbol{x},t,\boldsymbol{\xi},r)=j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big)
\end{aligned}
$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$, a.e. $(\boldsymbol{x},t)\in\Sigma_C$, $\xi_\nu=\boldsymbol{\xi}\cdot\boldsymbol{\nu}(\boldsymbol{x})$ and $\boldsymbol{\xi}_\tau=\boldsymbol{\xi}-\xi_\nu\boldsymbol{\nu}(\boldsymbol{x})$. Note that $\boldsymbol{\nu}$, $\boldsymbol{\tau}$, ξ_ν and $\boldsymbol{\xi}_\tau$ depend on $\boldsymbol{x}\in\Gamma_C$, but for simplicity we skip this dependence. It follows immediately that $j_i(\cdot,\cdot,\boldsymbol{\xi},r)$ are measurable on Σ_C for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$ and $j_i(\boldsymbol{x},t,\cdot,\cdot)$ are locally Lipschitz on $\mathbb{R}^{d+1}$ for a.e. $(\boldsymbol{x},t)\in\Sigma_C$, $i=1,2,3$. Moreover, from (26)(a), (27)(a) and (28)(a), we obtain $j(\cdot,\cdot,\mathbf{e}(\cdot))\in L^1(\Sigma_C)$, where $\mathbf{e}\in L^2(\Gamma_C;\mathbb{R}^{d+1})$ is given by

$$\mathbf{e}(\boldsymbol{x})=\big(e_1(\boldsymbol{x})\boldsymbol{\nu}+\left[\mathbf{e}_2(\boldsymbol{x})\right]_\tau,e_3(\boldsymbol{x})+\varphi_F(\boldsymbol{x})\big).$$

From the regularity of $j_\nu(\boldsymbol{x},t,\cdot)$, $j_\tau(\boldsymbol{x},t,\cdot)$ and $j_e(\boldsymbol{x},t,\cdot)$, we infer that $j_i(\boldsymbol{x},t,\cdot,\cdot)$ are regular for $i=1,2,3$. Using this regularity, by Proposition 5.6.33 of [5], we have

$$\partial j_i(\boldsymbol{x},t,\boldsymbol{\xi},r)\subseteq\partial_\xi j_i(\boldsymbol{x},t,\boldsymbol{\xi},r)\times\partial_r j_i(\boldsymbol{x},t,\boldsymbol{\xi},r)\tag{50}$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$, where ∂j_i denotes the generalized gradient of j_i with respect to $(\boldsymbol{\xi},r)$, $\partial_\xi j_i$ and $\partial_r j_i$ are the partial generalized gradients of $j_i(\boldsymbol{x},t,\cdot,r)$ and $j_i(\boldsymbol{x},t,\boldsymbol{\xi},\cdot)$, respectively, for $i=1,2,3$. Next, exploiting Proposition 2 of [11] and Proposition 5.6.23 of [5], from (50), we obtain

$$\partial j_1(\boldsymbol{x},t,\boldsymbol{\xi},r)\subseteq\partial j_\nu(\boldsymbol{x},t,\xi_\nu)\boldsymbol{\nu}\times\{0\},\tag{51}$$

$$\partial j_2(\boldsymbol{x},t,\boldsymbol{\xi},r)\subseteq\left[\partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_\tau)\right]_\tau\times\{0\},\tag{52}$$

$$\partial j_3(\boldsymbol{x},t,\boldsymbol{\xi},r)\subseteq\{\mathbf{0}\}\times\partial j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big),\tag{53}$$

$$\partial j(\boldsymbol{x},t,\boldsymbol{\xi},r)\subseteq\big(\partial j_\nu(\boldsymbol{x},t,\xi_\nu)\boldsymbol{\nu}+\left[\partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_\tau)\right]_\tau\big)\times\partial j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big)\tag{54}$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$. From (51)–(53) and the hypotheses (26)(c), (27)(c), (28)(c), we have

$$\big\|\partial j_1(\boldsymbol{x},t,\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}} \leqslant \big|\partial j_\nu(\boldsymbol{x},t,\xi_\nu)\big| \leqslant c_{0\nu}+c_{1\nu}|\xi_\nu| \leqslant c_{0\nu}+c_{1\nu}\big\|(\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}},$$
$$\big\|\partial j_2(\boldsymbol{x},t,\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}} \leqslant \big\|\partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_\tau)\big\|_{\mathbb{R}^d} \leqslant c_{0\tau}+c_{1\tau}\|\boldsymbol{\xi}_\tau\|_{\mathbb{R}^d} \leqslant c_{0\tau}+c_{1\tau}\big\|(\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}},$$
$$\big\|\partial j_3(\boldsymbol{x},t,\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}} \leqslant \big|\partial j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big)\big| \leqslant c_{0e}+c_{1e}\big|r-\varphi_F(\boldsymbol{x})\big| \leqslant c_{0e}+c_{1e}\big|\varphi_F(\boldsymbol{x})\big|+c_{1e}\big\|(\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}}$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$. From the above, we deduce that the function j given by (49) satisfies (3)(c) with

$$\bar{c}_0=\max\big\{c_{0\nu},c_{0\tau},c_{0e}+c_{1e}\|\varphi\|_{L^\infty(\Gamma_C)}\big\} \quad \text{and} \quad \bar{c}_1=\max\{c_{1\nu},c_{1\tau},c_{1e}\}.$$

Next, we check that the function j satisfies (3)(d). Let $(\boldsymbol{\zeta}_i,s_i)\in\partial j(\boldsymbol{x},t,\boldsymbol{\xi}_i,r_i)$ where $\boldsymbol{\zeta}_i$, $\boldsymbol{\xi}_i\in\mathbb{R}^d$, s_i, $r_i\in\mathbb{R}$, $i=1,2$. From the inclusion (54), we have $\boldsymbol{\zeta}_i=\bar{\zeta}_i\boldsymbol{\nu}+[\bar{\bar{\boldsymbol{\zeta}}}_i]_\tau$ with $\bar{\zeta}_i\in\mathbb{R}$, $\bar{\bar{\boldsymbol{\zeta}}}_i\in\mathbb{R}^d$, $\bar{\zeta}_i\in\partial j_\nu(\boldsymbol{x},t,\xi_\nu)$, $\bar{\bar{\boldsymbol{\zeta}}}_i\in\partial j_\tau(\boldsymbol{x},t,\boldsymbol{\xi}_\tau)$ and $s_i\in\partial j_e(\boldsymbol{x},t,r_i-\varphi_F(\boldsymbol{x}))$, $i=1,2$. Using (26)(d), (27)(d) and (28)(d) and the equality $\boldsymbol{\varrho}\cdot\boldsymbol{\xi}_\tau=\boldsymbol{\varrho}_\tau\cdot\boldsymbol{\xi}$ for $\boldsymbol{\varrho}$, $\boldsymbol{\xi}\in\mathbb{R}^d$, we obtain

$$\begin{aligned}
&\big((\boldsymbol{\zeta}_1,s_1)-(\boldsymbol{\zeta}_2,s_2)\big)\cdot\big((\boldsymbol{\xi}_1,r_1)-(\boldsymbol{\xi}_2,r_2)\big)\\
&\quad=(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2,s_1-s_2)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2,r_1-r_2)\\
&\quad=(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)+(s_1-s_2)(r_1-r_2)\\
&\quad=(\bar{\zeta}_1-\bar{\zeta}_2)\boldsymbol{\nu}\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)+\big([\bar{\bar{\boldsymbol{\zeta}}}_1]_\tau-[\bar{\bar{\boldsymbol{\zeta}}}_2]_\tau\big)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)+(s_1-s_2)(r_1-r_2)\\
&\quad=(\bar{\zeta}_1-\bar{\zeta}_2)(\xi_{1\nu}-\xi_{2\nu})+(\bar{\bar{\boldsymbol{\zeta}}}_1-\bar{\bar{\boldsymbol{\zeta}}}_2)\cdot(\boldsymbol{\xi}_{1\tau}-\boldsymbol{\xi}_{2\tau})+(s_1-s_2)(r_1-r_2)\\
&\quad\geqslant -m_\nu|\xi_{1\nu}-\xi_{2\nu}|^2-m_\tau\|\boldsymbol{\xi}_{1\tau}-\boldsymbol{\xi}_{2\tau}\|^2_{\mathbb{R}^d}-m_e|r_1-r_2|^2\\
&\quad\geqslant -\max\{m_\nu,m_\tau,m_e\}\big\|(\boldsymbol{\xi}_1,r_1)-(\boldsymbol{\xi}_2,r_2)\big\|^2_{\mathbb{R}^{d+1}}.
\end{aligned}$$

This proves (3)(d) with $m_2=\max\{m_\nu,m_\tau,m_e\}\geqslant 0$.

Subsequently, using the definition of the generalized directional derivative of $j_i(\boldsymbol{x},t,\cdot,\cdot)$, $i=1,2,3$, from Proposition 2 of [11], we have

$$\begin{aligned}
&j_1^0(\boldsymbol{x},t,\boldsymbol{\xi},r;\boldsymbol{\varrho},s)\leqslant j_\nu^0(\boldsymbol{x},t,\xi_\nu;\varrho_\nu),\\
&j_2^0(\boldsymbol{x},t,\boldsymbol{\xi},r;\boldsymbol{\varrho},s)\leqslant j_\tau^0(\boldsymbol{x},t,\boldsymbol{\xi}_\tau;\boldsymbol{\varrho}_\tau),\\
&j_3^0(\boldsymbol{x},t,\boldsymbol{\xi},r;\boldsymbol{\varrho},s)\leqslant j_e^0\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x});s\big)
\end{aligned}$$

for all $(\boldsymbol{\xi},r)$, $(\boldsymbol{\varrho},s)\in\mathbb{R}^{d+1}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$. Hence, using Proposition 2.3.3 of [4], we obtain

$$\begin{aligned}
j^0(\boldsymbol{x},t,\boldsymbol{\xi},r;-\boldsymbol{\xi},-r)&\leqslant j_1^0(\boldsymbol{x},t,\boldsymbol{\xi},r;-\boldsymbol{\xi},-r)+j_2^0(\boldsymbol{x},t,\boldsymbol{\xi},r;-\boldsymbol{\xi},-r)+j_3^0(\boldsymbol{x},t,\boldsymbol{\xi},r;-\boldsymbol{\xi},-r)\\
&\leqslant j_\nu^0(\boldsymbol{x},t,\xi_\nu;-\xi_\nu)+j_\tau^0(\boldsymbol{x},t,\boldsymbol{\xi}_\tau;-\boldsymbol{\xi}_\tau)+j_e^0\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x});-r\big)
\end{aligned}\tag{55}$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$. From the subadditivity of the function $j_e^0(\boldsymbol{x},t,r;\cdot)$ for all $r\in\mathbb{R}$, a.e. $(\boldsymbol{x},t)\in\Sigma_C$, Proposition 2.1.2(b) of [4] and (28)(c), (e), we have

$$\begin{aligned}
j_e^0\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x});-r\big)&\leqslant j_e^0\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x});-\big(r-\varphi_F(\boldsymbol{x})\big)\big)+j_e^0\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x});-\varphi_F(\boldsymbol{x})\big)\\
&\leqslant d_e\big(1+\big|r-\varphi_F(\boldsymbol{x})\big|\big)+\max\big\{\eta\big(-\varphi_F(\boldsymbol{x})\big)\ \big|\ \eta\in\partial j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big)\big\}\\
&\leqslant d_e\big(1+|r|+\big|\varphi_F(\boldsymbol{x})\big|\big)+\big|\varphi_F(\boldsymbol{x})\big|\big|\partial j_e\big(\boldsymbol{x},t,r-\varphi_F(\boldsymbol{x})\big)\big|\leqslant\widetilde{d}\big(1+|r|\big)
\end{aligned}$$

for all $r\in\mathbb{R}$ and a.e. $(\boldsymbol{x},t)\in\Sigma_C$ with $\widetilde{d}>0$. Therefore, using (26)(e), (27)(e) and (55), it follows that

$$j^0(\boldsymbol{x},t,\boldsymbol{\xi},r;-\boldsymbol{\xi},-r)\leqslant d_\nu\big(1+|\xi_\nu|\big)+d_\tau\big(1+\|\boldsymbol{\xi}_\tau\|_{\mathbb{R}^d}\big)+\widetilde{d}\big(1+|r|\big)\leqslant d\big(1+\big\|(\boldsymbol{\xi},r)\big\|_{\mathbb{R}^{d+1}}\big)$$

for all $(\boldsymbol{\xi},r)\in\mathbb{R}^{d+1}$, a.e. $(\boldsymbol{x},t)\in\Sigma_C$ with $d>0$. It follows from here that (3)(e) holds and we conclude that the function j given by (49) satisfies (3). Furthermore, the regularity hypotheses (44) on either j_ν, j_τ, j_e or $-j_\nu$, $-j_\tau$ $-j_e$ imply the regularity of either j or $-j$, respectively.

Finally, we consider the element $f=(\boldsymbol{f},q)$ and we note that assumptions (29), (30) imply that $f\in\mathcal{Y}^*$.

We are now in a position to apply Theorem 1. From this theorem, we know that the history-dependent hemivariational inequality

$$\big\langle A\big(t,y(t)\big),z\big\rangle_{Y^*\times Y}+\big\langle\mathcal{S}y(t),z\big\rangle_{Y^*\times Y}+\int_{\Gamma_C} j^0\big(t,\gamma y(t);\gamma z\big)\,d\Gamma\geqslant\big\langle f(t),z\big\rangle_{Y^*\times Y}\quad\text{for all } z\in Y,\ \text{a.e. } t\in(0,T)\tag{56}$$

has a unique solution $y=(\boldsymbol{w},\varphi)\in\mathcal{Y}$. It remains to show that from (56) we retrieve the coupled system (42)–(43). Choosing $z=(\boldsymbol{v},0)\in Y$ with $\boldsymbol{v}\in V$ in (56), we obtain the inequality (42) and taking $z=(\boldsymbol{0},\psi)\in Y$ with $\psi\in\Phi$, we get (43). Therefore, we conclude that (56) is equivalent to the system (42)–(43). It follows that the couple of functions $(\boldsymbol{w},\varphi)$ represents the unique solution to Problem P_V, which completes the proof of the theorem. □

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